ON THE ACOUSTIC THEORY OF SPALLING

(OB AKUSTICHESKOI TEORII OTKOLA)

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Lenskii [1] has shown that it is possible to solve the problem of spalling [scabbing] by the acoustic approach. A solution of the above problem is constructed below with the aid of the method of incomplete separation of variables [1, 2] and operational calculus.

Assume that at time $t = 0$, a force $P(t)$ is applied at some point of the surface of a plate, $z = 0$, and that the lower surface of the plate is free of stress.

At first, a solution for a unit step force

$$\delta(t) = \begin{cases} 0 & (t < 0) \\ 1 & (t \geqslant 0) \end{cases}$$

is constructed, from which it is easy to go over to an arbitrary time variation of the force.

In fact, if $\sigma^*(r, z, t)$ gives the stress distribution under the action of a unit step force, then the derivative $\sigma_t^*(r, z, t)$ will correspond to the stresses caused by a unit impulse. The stresses due to the force $P(t)$ ($P(t) \neq 0$ for $0 < t < \epsilon$, $P(t) \equiv 0$ for $t < 0$ and $t > \epsilon$) will be determined by the formula

$$\sigma(r, z, t) = \int_0^{\varepsilon} P(\tau)\, \sigma_t^*(r, z, t - \tau)\, d\tau$$

Integrating by parts and changing the variable of integration, the last formula can be written as follows:

$$\sigma(r, z, t) = P(0)\, \sigma^*(r, z, t) + \int_{t-\varepsilon}^{t} P'(t - t_1)\, \sigma^*(r, z, t_1)\, dt_1 \qquad (1)$$

An analogous formula also holds for the displacements.

Assume at first, that the upper surface of the plate is acted upon by a pressure distributed along the radius as follows:

$$\sigma(r, 0, t) = -\frac{1}{2\pi} \frac{n^2}{(1 + n^2R^2)^{3/2}} \delta(t)$$

In the limit as $n \to \infty$ the unit step force is obtained.

The solution of the wave equation for the displacement potential (where c is the speed of sound)

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}$$

with zero initial conditions and the specified boundary conditions will be found, assuming

$$\varphi(r, z, t) = \int_0^\infty \Phi(k, z, t) I_0(kr)\, dk, \qquad \overline{\Phi}(r, z, s) = \int_0^\infty \Phi(k, z, t) e^{-st}\, dt$$

An ordinary differential equation for the function $\Phi(k, z, s)$ is obtained. Its solution, satisfying the boundary conditions, is

$$\overline{\Phi}(k, z, s) = \frac{c^2}{2\pi\lambda} ke^{-k/n} \frac{\overline{\delta}(s)}{s^2}\left[-\sum_{m=0}^{\infty} e^{-\theta_m(k, z, s)} + \sum_{m=1}^{\infty} e^{-\vartheta_m(k, z, s)}\right]$$

$$\theta_m(k, z, s) = (2mh + z)\sqrt{k^2 + \frac{z^2}{c^2}}, \qquad \vartheta_m(k, z, s) = (2mh - z)\sqrt{k^2 + \frac{s^2}{c^2}}$$

where h is the thickness of the plate, λ is Lame's constant, and $\overline{\delta}(s) = 1/s$.

Applying the inversion theorem, the displacement potential is found to be:

$$\varphi(r, z, t) = \frac{c^2}{2\pi\lambda}\left\{-\sum_{m=0}^{\infty}\int_0^\infty ke^{-k/n}\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\overline{\delta}(s)}{s^2} e^{-\theta_m(k, z, s)+st}\, ds\right] I_0(kr)\, dk + \right.$$

$$\left. + \sum_{m=0}^{\infty}\int_0^\infty ke^{-k/n}\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\overline{\delta}(s)}{s^2} e^{-\vartheta_m(k, z, s)+st}\, ds\right] I_0(kr)\, dk\right\}$$

It is known from operational calculus that each term of the series shown starts to differ from zero only at the time when the following inequalities begin to be satisfied:

$$-2mh - z + t \geqslant 0, \qquad \text{or} \qquad -2mh + z + t \geqslant 0$$

The incident wave and waves reflected from the upper surface appear in the first sum; waves reflected from the lower surface appear in the second sum.

After changing the variable of integration in the contour integrals to $\zeta = s/k$, the stress $\sigma^*(r, z, t)$ can be represented in the following fashion:

$$\sigma^*(r, z, t) = \frac{\lambda}{c^2}\frac{\partial^2\varphi}{\partial t^2} =$$

$$= \frac{1}{2\pi}\left\{-\sum_{m=0}^{\infty}\int_0^{\infty} k^2 e^{-k/n}\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\bar{\delta}(k\zeta)\, e^{-\theta_m(k, z, k\zeta)+k\zeta t}\, d\zeta\right] I_0(kr)\, dk +\right.$$

$$\left. + \sum_{m=1}^{\infty}\int_0^{\infty} k^2 e^{-k/n}\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\bar{\delta}(k\zeta)\, e^{-\vartheta_m(k, z, k\zeta)+k\zeta t}\, d\xi\right] I_0(kr)\, dk\right.$$

It is of interest for the solution of the problem of spalling to determine the stresses along the axis ($r = 0$) for two waves: the incident wave and the wave reflected from the lower surface of the plate.

One can go from the straight line, parallel to the imaginary axis in the ξ plane, along which the contour integral is taken, by continuous deformation without cutting across singular points of the integrands, to such contours l, along which it is possible to interchange the orders of integration and to carry out the integration with respect to the variable k [3]. Carrying out the operations described above, and letting $n = \infty$, one obtains for the incident wave

$$\sigma_I^*(0, z, t) = -\frac{1}{2\pi}\frac{1}{2\pi i}\int_l \frac{1}{\xi\left[z\sqrt{1+\frac{\xi^2}{c^2}} - \xi t\right]^2}\, d\xi$$

and for the reflected wave

$$\sigma_{II}^*(0, z, t) = \frac{1}{2\pi}\frac{1}{2\pi i}\int_l \frac{1}{\xi\left[z(2h-z)\sqrt{1+\frac{\xi^2}{c^2}} - \xi t\right]^2}\, d\xi$$

The integrands in the last relations have one real pole each, to the right of the contour of integration. Application of the residue theorem yields

$$\sigma_I^*(0, z, t) = -\frac{1}{2\pi}\frac{1}{z^2}\delta\left(t - \frac{z}{c}\right), \qquad \sigma_{II}(0, z, t) = \frac{1}{2\pi}\frac{1}{(2h-z)^2}\delta\left(t - \frac{2h-z}{c}\right)$$

With the aid of formula (1), the stress due to the action of P can be found ($P(t) \neq 0$ for $0 \leqslant t \leqslant \epsilon$; $P(t) \equiv 0$ for $t < 0$ and $t > \epsilon$):

$$\sigma_I(0, z, t) = -\frac{1}{2\pi}\frac{1}{z^2}P\left(t - \frac{z}{c}\right), \qquad \sigma_{II}(0, z, t) = \frac{1}{2\pi}\frac{1}{(2h-z)^2}P\left(t - \frac{2h-z}{c}\right)$$

In contrast to the result given by formula (3) in Ref. [1] one always gets a compressive stress at the head of the incident wave front at an arbitrary z:

$$\sigma_{II}^{fr} = -\frac{1}{2\pi}\cdot\frac{1}{z^2}P(0)$$

Therefore, in the case investigated here, face spalling cannot occur.

As far as rear [back] spalling is concerned, it can be explained by the appearing of tensile stresses when the wave is reflected from the lower surface of the plate.

With an arbitrary law of variation of the force $P(t)$, the tensile stresses will sometimes reach destructive magnitudes for the first time at the front of the reflected wave. This will take place, for instance, when $P(0)$ is the largest value. In this case, in order to determine the depth of rear spalling z_0, the stress at the head of the reflected wave front will be equated to the value of the destructive stress σ_p.

The following equation serves for the determination of z_0:

$$-\frac{1}{z_0^2} P\left(\frac{2h-2z_0}{c}\right)+\frac{1}{(2h-z_0)^2} P(0)=2\pi\sigma_p$$

If $P(t)$ varies with time according to a triangular law

$$P(t)=\begin{cases}0 & t<0\\ P(0)(1-t/\varepsilon) & 0\leqslant t\leqslant\varepsilon\\ 0 & t>\varepsilon\end{cases}$$

then the equation for the determination of z_0 simplifies as follows:

$$-\frac{1}{z_0^2}\left(1-\frac{2(h-z_0)}{\varepsilon c}\right)+\frac{1}{(2h-z_0)^2}=\frac{2\pi}{P(0)}\sigma_p.$$

The solution obtained above is different from the solution in [1]. This is explained by the fact that in Ref. [1] a solution of the wave equation is sought, which satisfies zero initial conditions, boundary conditions on the surface $z = 0$ (except for the point $r = 0$), and retains an arbitrary constant in the form of a coefficient. This constant is determined from the condition that $P(t)$ be the limit of the resultant stresses on the surface of a hemisphere with its center at the point $r = 0$, $z = 0$, whose radius tends towards zero. The last condition appears to be superfluous and does not ensure a correct choice of solution.

The result shows that the solution does not satisfy the other necessary condition, which requires that the limit of the resultant inertia forces in the direction of the z axis be zero when the radius of the hemisphere, whose center is at the origin, tends towards zero. Besides, it is found that the resultant of the inertia forces in the radial direction, for that part of the hemisphere, which is cut out by two planes passing through the axis of symmetry, is infinitely large for an arbitrary finite radius.

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